Iterative Structure Within The Five-Particle Two-Loop Amplitude

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Abstract

We find an unexpected iterative structure within the two-loop five-gluon amplitude in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Specifically, we show that a subset of diagrams contributing to the full amplitude, including a two-loop pentagon-box integral with nontrivial dependence on five kinematical variables, satisfies an iterative relation in terms of one-loop scalar box diagrams. The implications of this result for the possible iterative structure of the full two-loop amplitude are discussed.

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I. INTRODUCTION

Scattering amplitudes in gauge theory exhibit simplicity that is not manifest from Feynman diagrams [1]. This simplicity is even more striking in the case of maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory (MSYM).

One of the fascinating properties of MSYM is the possible presence of iterative structures relating amplitudes at different orders in perturbation theory. In [2] Anastasiou, Bern, Dixon and Kosower (ABDK) suggested that the planar two-loop n-gluon maximally helicity violating (MHV) amplitude obeys the iteration

$$M_n^{(2)}(\epsilon) = \frac{1}{2} \left(M_n^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon), \tag{1}$$

where $M_n^{(L)}(\epsilon) = A_n^{(L)}(\epsilon)/A_n^{(0)}$ is the ratio of the L-loop amplitude (evaluated in $D = 4 - 2\epsilon$ dimensions) to the corresponding tree-level amplitude, and

$$f^{(2)}(\epsilon) \equiv -\zeta(2) - \zeta(3)\epsilon - \zeta(4)\epsilon^2. \tag{2}$$

This proposal was inspired by the fact that the collinear [2, 3, 4, 5] and infrared singular [6, 7, 8] pieces of $M_n^{(2)}(\epsilon)$ are known to satisfy an iterative relation of this form. The statement that (1) actually holds for the full two-loop amplitude $M_n^{(2)}(\epsilon)$ is the content of the ABDK conjecture, which at present has been explicitly checked only for the $n = 4$ gluon amplitude [2]. A similar iterative relation at three loops was recently proven in [9], also for $n = 4$.

The four-gluon amplitude is very special in the sense that it is a nontrivial[1] function of a single dimensionless variable $x = t/s$. The $n = 5$ case is much more complicated due to the presence of several independent kinematical invariants. This is also the reason why one would expect a richer, less rigid structure compared to the $n = 4$ case.

Although the two-loop five-gluon amplitude is not known, a conjecture for it has been given in [10]. The conjecture for $M_5^{(2)}(\epsilon)$ has the interesting property that contains two classes of terms,

$$M_5^{(2)}(\epsilon) = V_5^{(2)}(\epsilon) + W_5^{(2)}(\epsilon), \tag{3}$$

where $V_5^{(2)}(\epsilon)$ is parity even and $W_5^{(2)}(\epsilon)$ is parity odd. This is in contrast to the four-gluon amplitude $M_4^{(2)}(\epsilon)$, which is wholly parity even. In this paper we evaluate $V_5^{(2)}(\epsilon)$

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[1] There is a 'trivial' overall factor of $(st)^{-\epsilon}$ in each term in equation (1).

explicitly through $\mathcal{O}(\epsilon^{-1})$ and numerically[2] at $\mathcal{O}(\epsilon^0)$. As one step in this calculation, we present an explicit (through $\mathcal{O}(\epsilon^{-1})$) formula for a two-loop pentagon-box integral. To our knowledge this is the first appearance in the literature of a two-loop integral depending on five kinematical invariants. In performing this calculation we have benefited from a program recently developed by Czakon [11] which greatly facilitates the manipulation and numerical evaluation of Mellin-Barnes integrals.

In order to check the relation (1) for $n = 5$, it is necessary to know the one-loop amplitude $M_5^{(1)}(\epsilon)$ through $\mathcal{O}(\epsilon^2)$. Through $\mathcal{O}(\epsilon^0)$, it can be written as a linear combination of one-loop scalar box integrals, but starting at $\mathcal{O}(\epsilon)$ extra terms appear which can be expressed in terms of pentagon integrals in $D = 6 - 2\epsilon$ dimensions [12, 13, 14]. To our knowledge, the contribution of these extra pieces to $M_5^{(1)}(\epsilon)$ has not been explicitly computed. We therefore decompose

$$M_5^{(1)}(\epsilon) = V_5^{(1)}(\epsilon) + W_5^{(1)}(\epsilon), \qquad W_5^{(1)}(\epsilon) = \mathcal{O}(\epsilon), \tag{4}$$

where $V_5^{(1)}(\epsilon)$ consists of the one-loop scalar box terms and $W_5^{(1)}(\epsilon)$ contains the (currently unknown) extra pieces from the pentagon integrals.

In this paper we prove the remarkable fact that the $V_5^{(L)}(\epsilon)$ pieces alone satisfy the ABDK relation (1), i.e. we prove that

$$V_5^{(2)}(\epsilon) = \frac{1}{2} \left(V_5^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) V_5^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon). \tag{5}$$

This paper is organized as follows: In section II we define the various integrals which are studied in this paper and review the proposal [10] for the two-loop give-gluon amplitude. In section III we provide some details of the proof of our main result (5), postponing most technical details to the appendix. In section IV we use double-double unitarity cuts to show that $M_5^{(2)}(\epsilon)$ must contain the term $W_5^{(2)}(\epsilon)$, and we demonstrate the intriguing fact that this term has very mild IR behaviour, $W_5^{(2)}(\epsilon) = \mathcal{O}(\epsilon^{-1})$. In section V we conclude with a discussion of our results and their possible implications for the iterative structure of the full two-loop amplitude.

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[2] More specifically, it is the relation (5) that we check numerically at $\mathcal{O}(\epsilon^0)$. After this is done, it can of course be turned around as in section III.B and used to construct an explicit formula for $V_5^{(2)}(\epsilon)$ through $\mathcal{O}(\epsilon^0)$.
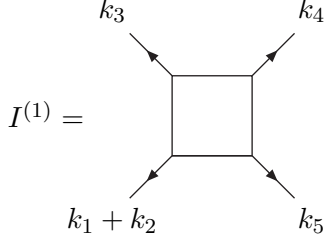
$$I^{(1)} =$$

k_3 k_4

$k_1 + k_2$ k_5

FIG. 1: The one-loop one-mass scalar box integral. See appendix A for details.

II. FIVE-GLUON AMPLITUDES AT ONE AND TWO LOOPS

In this section we present formulas for the planar five-gluon amplitudes that enter into the main result (5). Recall that for MHV amplitudes it is very useful to normalize loop amplitudes by the corresponding tree-level amplitudes. Let us denote by $M_5^{(L)}(\epsilon)$ the normalized L-loop amplitude $A_5^{(L)}(\epsilon)/A_5^{(0)}$. We use the notation

$$s_i \equiv -(k_i + k_{i+1})^2. \tag{6}$$

The one-loop five-gluon amplitude is known to be [12]

$$M_5^{(1)}(\epsilon) = -\frac{1}{4} \sum_{\text{cyclic}} \left[s_3 s_4 I^{(1)}(\epsilon) \right] + \mathcal{O}(\epsilon), \tag{7}$$

where $I^{(1)}(\epsilon)$ is the one-loop scalar box integral shown in Figure 1 and the sum is taken over the five cyclic permutations of the external momenta. The missing $\mathcal{O}(\epsilon)$ terms in (7), which involve $D = 6 - 2\epsilon$ pentagon integrals [14], are not explicitly known. We therefore define

$$V_5^{(1)}(\epsilon) \equiv -\frac{1}{4} \sum_{\text{cyclic}} \left[s_3 s_4 I^{(1)}(\epsilon) \right] \tag{8}$$

with the understanding that $V_5^{(1)}(\epsilon)$ and $M_5^{(1)}(\epsilon)$ differ starting at $\mathcal{O}(\epsilon)$. This difference is defined to be $W_5^{(1)}(\epsilon) \equiv M_5^{(1)}(\epsilon) - V_5^{(1)}(\epsilon)$.

At two loops, it has been conjectured [10] that[3]

$$M_5^{(2)}(\epsilon) = -\frac{1}{8} \sum_{\text{cyclic}} \left[s_3 s_4^2 I^{(2)a}(\epsilon) + s_3^2 s_4 I^{(2)b}(\epsilon) + s_1 s_3 s_4 I^{(2)d}(\epsilon) + s_1 I^{(2)e}(\epsilon) \right], \tag{9}$$

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[3] Note that we have translated this expression into the more modern normalization conventions of [2, 9].

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where the various integrals are defined in Figure 2. This formula for $M_5^{(2)}(\epsilon)$ can clearly be written as a sum of two kinds of terms,

$$V_5^{(2)}(\epsilon) = -\frac{1}{8} \sum_{\text{cyclic}} \left[s_3 s_4^2 I^{(2)a}(\epsilon) + s_3^2 s_4 I^{(2)b}(\epsilon) + s_1 s_3 s_4 I^{(2)d}(\epsilon) \right], \quad (10)$$

$$W_5^{(2)}(\epsilon) = -\frac{1}{8} \sum_{\text{cyclic}} \left[s_1 I^{(2)e}(\epsilon) \right], \quad (11)$$

which are respectively parity even and parity odd. The results of this paper provide very strong evidence in support of the $V_5^{(2)}(\epsilon)$ part of this conjecture, and in section 3 we show that the $W_5^{(2)}(\epsilon)$ term must be present in $M_5^{(2)}(\epsilon)$ as well.

III.   PROOF OF THE MAIN RESULT

In this section we present our proof that $V_5^{(1)}(\epsilon)$ and $V_5^{(2)}(\epsilon)$, defined in (8) and (10) respectively, satisfy the ABDK relation (5)

$$V_5^{(2)}(\epsilon) = \frac{1}{2} \left(V_5^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) V_5^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon). \quad (12)$$

Our method is similar to that used in [15] to study the four-gluon amplitude $M_4^{(2)}(\epsilon)$ in that we are able to check (5) without the need to fully evaluate any of the loop integrals in terms of harmonic polylogarithm functions[4]. This is done by deriving the necessary identities between various Mellin-Barnes integrals 'under the integral sign'.

We begin in subsection III.A by verifying (5) explicitly through $\mathcal{O}(\epsilon^{-1})$. That this works is already highly nontrivial for two reasons. First of all is the fact that the ansatz (9) for $M_5^{(2)}(\epsilon)$ has not been completely proven, although our results obviously provide strong evidence in its favor. Second is the fact that, even if one assumes that the $M_5^{(L)}(\epsilon)$ satisfy (1), there is no obvious reason why this should imply that the $V_5^{(L)}(\epsilon)$ satisfy (5). The difference between (1) and (5) first shows up at $\mathcal{O}(\epsilon^{-1})$ because (as we show below) $W_5^{(2)}(\epsilon)$ starts contributing to the left-hand side of (5) at this order, and $W_5^{(1)}(\epsilon)$ starts contributing to the right-hand side of (5) also at this order.

It seems a miracle that (5) holds at $\mathcal{O}(\epsilon^{-1})$ even though so many things might have spoiled it. In subsection III.B we show that this miracle persists through $\mathcal{O}(\epsilon^0)$. It seems clear that

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[4] Although it is not necessary for our proof, we nevertheless present in appendix A several explicit formulas in terms of polylogs, in particular for the integrals $I^{(2)c}$ and $I^{(2)d}$ which are new to the literature.

$$I^{(2)a} =$$

$$I^{(2)b} =$$

$$I^{(2)c} =$$

$$I^{(2)d} = (q - k_1)^2 \times$$

$$I^{(2)e} = (q - k_1)^2 \, \mathrm{tr}[\gamma_5 \, (\slashed{k}_1 + \slashed{k}_2) \, \slashed{k}_3 \, \slashed{q} \, \slashed{k}_5] \times$$
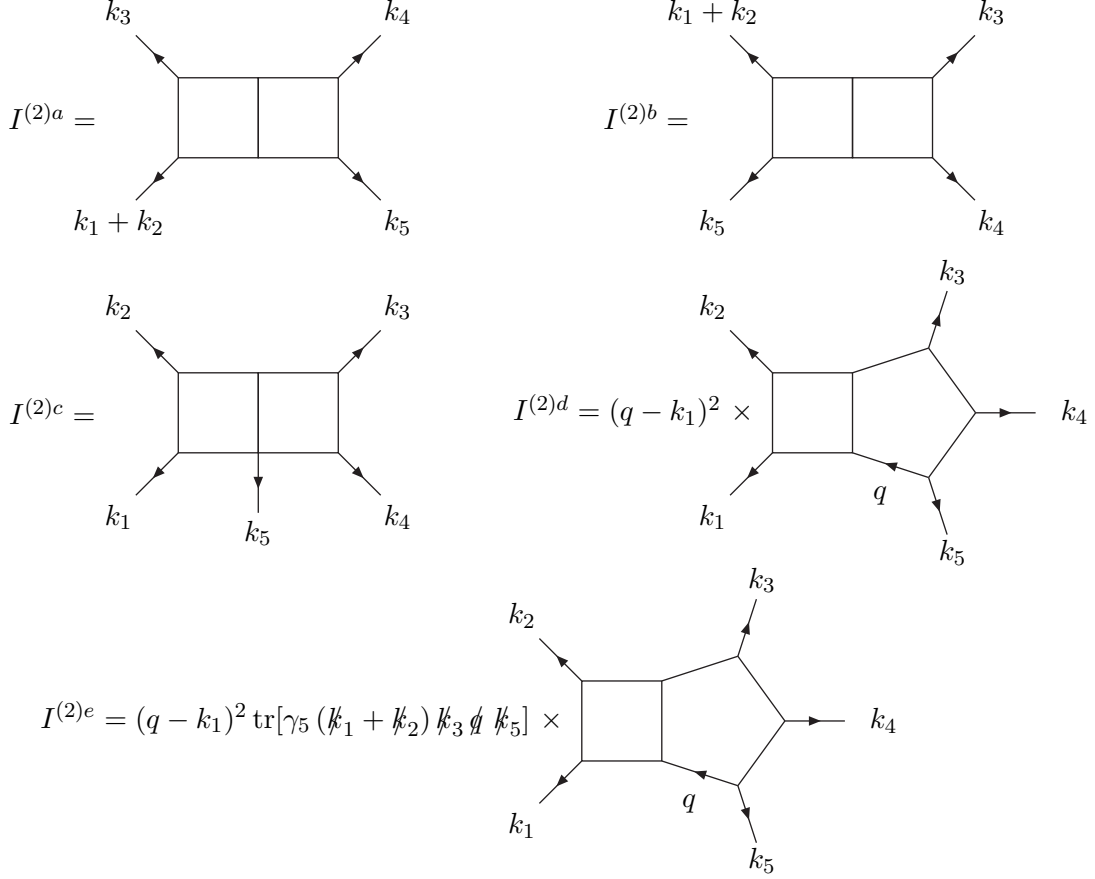
FIG. 2: Cast of characters: here we define the five two-loop integrals which appear in this paper. All figures refer to the corresponding diagrams in a scalar field theory, i.e., a collection of propagators integrated over the loop momenta. The integrals $I^{(2)d}$ and $I^{(2)e}$ are multiplied by factors which depend on one of the loop momenta, as indicated explicitly above. These factors are meant to be included in the numerators of the corresponding scalar integrals, transforming them into tensor integrals. See appendix A for further details.

the $\mathcal{O}(\epsilon^0)$ terms in (5) could, in principle, also be checked without fully evaluating the integrations explicitly, but it would require a larger number of more complicated identities. For the purpose of this paper, which is to point out an unexpected iterative structure within the full amplitude, we are content to perform the final step at $\mathcal{O}(\epsilon^0)$ numerically, via a robust procedure described below.

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A. The ABDK Relation for $V_5^{(L)}(\epsilon)$ Through $\mathcal{O}(\epsilon^{-1})$

Explicit formulas for the integrals $I^{(1)}(\epsilon)$ and $I^{(2)a}(\epsilon)$ have appeared in the literature [14, 16]. We have evaluated the new integral $I^{(2)d}(\epsilon)$ explicitly through $\mathcal{O}(\epsilon^{-1})$. The results for these integrals are summarized in appendix A, where they are expressed in terms of several basic Mellin-Barnes-type integrals which we call $A(a)$, $A_j(a)$ and $F(a,b)$. The A functions are defined by the integrals

$$A(a) = \int \frac{dz}{2\pi i}\, a^z\, \Gamma^2(-z)\Gamma(z)\Gamma(1+z), \tag{13}$$

$$A_1(a) = \int \frac{dz}{2\pi i}\, a^z\, \Gamma^2(-z)\Gamma(z)\Gamma(1+z)(\psi(z)+\gamma), \tag{14}$$

$$A_2(a) = \int \frac{dz}{2\pi i}\, a^z\, \Gamma^2(-z)\Gamma(z)\Gamma(1+z)(\psi(-z)+\gamma), \tag{15}$$

$$A_3(a) = \int \frac{dz}{2\pi i}\, a^z\, \Gamma^2(-z)\Gamma(z)\Gamma(1+z)\frac{1}{z}, \tag{16}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ and in each case z is integrated from $-i\infty$ to $+i\infty$ along a contour which intersects the real axis between -1 and 0. The function $F(a,b)$ is given by a more complicated double integral defined in appendix A. Here we only need to use the fact that it is symmetric in a and b. Explicit formulas for all of these functions in terms of harmonic polylogarithms are given in appendix A.

If we plug the expressions given in appendix A for the various integrals into (5), we find that the $\mathcal{O}(\epsilon^{-4})$ through $\mathcal{O}(\epsilon^{-2})$ pieces cancel easily, leaving just

$$V_5^{(2)}(\epsilon) - \left[\frac{1}{2}\left(V_5^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)V_5^{(1)}(2\epsilon)\right]$$

$$= \frac{1}{\epsilon}\sum_{i=1}^{5}\left[\frac{1}{4}A_1\left(\frac{s_i}{s_{i-2}}\right) - \frac{1}{2}A_1\left(\frac{s_i}{s_{i+2}}\right) - \frac{1}{4}A_2\left(\frac{s_i}{s_{i-2}}\right) + \frac{1}{2}A_2\left(\frac{s_i}{s_{i+2}}\right) - \frac{1}{8}A_3\left(\frac{s_i}{s_{i-2}}\right)\right.$$

$$\left. - \frac{1}{4}\ln\left(\frac{s_i}{s_{i+2}}\right)A\left(\frac{s_i}{s_{i+2}}\right) + \frac{1}{8}\ln\left(\frac{s_i}{s_{i-2}}\right)A\left(\frac{s_i}{s_{i-2}}\right) - \frac{1}{24}\ln^3\left(\frac{s_i}{s_{i-2}}\right)\right] + \mathcal{O}(\epsilon^0). \tag{17}$$

Interestingly, all of the $F(a,b)$ terms automatically drop out of this expression due to the symmetry $F(a,b) = F(b,a)$. The remaining terms on the last two lines of (17) can be seen to vanish with the help of the identities

$$A(a) + A(1/a) = -\frac{1}{2}\ln^2(a) - \frac{\pi^2}{3}, \tag{18}$$

$$A_1(a) - A_2(a) + \frac{1}{2}A_3(a) = -\frac{1}{2}\ln(a)A(a), \tag{19}$$

$$A_1(1/a) - A_2(1/a) + \frac{1}{2}A_3(a) = -\frac{1}{2}\ln(a)A(a) - \frac{\pi^2}{3}\ln(a) - \frac{1}{3}\ln^3(a), \tag{20}$$

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which can be derived by applying various tricks directly to the definitions (13)–(16). This illustrates our point that it is possible to verify (17) without the need to fully evaluate loop integrals in terms of harmonic polylogarithms.

An interesting fact, addressed in section V below, is that the integral $I^{(2)c}$ does not seem to appear in $V_5^{(2)}(\epsilon)$ (in the chosen basis of integrals). Suppose we make a more general ansatz

$$-\frac{1}{8}\sum_{\text{cyclic}}\left[f_a s_3 s_4^2 I^{(2)a}(\epsilon) + f_b s_3^2 s_4 I^{(2)b}(\epsilon) + f_c s_1 s_2 s_3 I^{(2)c}(\epsilon) + f_d s_1 s_3 s_4 I^{(2)d}(\epsilon)\right] \tag{21}$$

and ask whether there exist numbers $f_a, \cdots, f_d$ such that (21) satisfies (5). At order $\mathcal{O}(\epsilon^{-4})$ and $\mathcal{O}(\epsilon^{-3})$ the only constraint is that the f's should satisfy

$$f_a + f_b + \frac{9}{4}f_c + 3f_d = 5. \tag{22}$$

However, at $\mathcal{O}(\epsilon^{-2})$ the functional dependence of the various integrals on the s_i is much more complicated, and it is easy to see that the unique solution enabling (21) to satisfy (5) even through $\mathcal{O}(\epsilon^{-2})$ is

$$f_a = f_b = f_d = 1, \qquad f_c = 0. \tag{23}$$

B. The ABDK Relation at $\mathcal{O}(\epsilon^0)$ and The Finite Remainder

Having demonstrated that $V^{(2)}(\epsilon)$ satisfies the ABDK relation through $\mathcal{O}(\epsilon^{-1})$, let us now turn our attention to the $\mathcal{O}(\epsilon^0)$ piece. In principle one could continue as in the previous subsection by identifying the basic integrals which emerge from the Mellin-Barnes representation and then working out the necessary identities relating them to each other. However, as mentioned above, it suffices for our purpose here to perform this final step numerically.

A very efficient program for numerical evaluation of Mellin-Barnes integrals has recently been developed by Czakon [11] (see also [17]). We repeatedly evaluated each term in (5) separately, for randomly generated values of the kinematical variables s_i, and always found that although each term in (5) is generically $\mathcal{O}(1)$, the left-hand and the sum of the terms on the right-hand side conspired to agree with each other to within the precision of the numerical integrations ($\sim 10^{-9}$). This highly nontrivial conspiracy leaves us with no doubt that (5) is correct.

A robust way to distill the information contained in (5) at $\mathcal{O}(\epsilon^0)$ is to study the so-called finite remainder of $V^{(2)}(\epsilon)$. The finite remainder of an arbitrary L-loop n-gluon amplitude was defined in [9]. For the case at hand, we construct the quantities

$$\hat{I}^{(1)}(\epsilon) \equiv -\frac{1}{2\epsilon^2} \sum_{i=1}^{5} (s_i)^{-\epsilon}, \tag{24}$$

$$\hat{I}^{(2)}(\epsilon) \equiv -\frac{1}{2} \left(\hat{I}^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) \hat{I}^{(1)}(2\epsilon), \tag{25}$$

in terms of which the finite remainders for $V_5^{(L)}(\epsilon)$ are defined by

$$F_5^{(1)}(\epsilon) \equiv V_5^{(1)}(\epsilon) - \hat{I}^{(1)}(\epsilon), \tag{26}$$

$$F_5^{(2)}(\epsilon) \equiv V_5^{(2)}(\epsilon) - \left[\hat{I}^{(2)}(\epsilon) + \hat{I}^{(1)}(\epsilon) V_5^{(1)}(\epsilon) \right]. \tag{27}$$

Of course [9] defined this quantity with the full amplitudes $M_5^{(L)}(\epsilon)$ instead of just the pieces $V_5^{(L)}(\epsilon)$ we are using here. Nevertheless, the analysis of that paper proves that if $V_5^{(2)}(\epsilon)$ satisfies (5), then $F_5^{(2)}(\epsilon)$ must be given by

$$F_5^{(2)}(\epsilon) = \frac{1}{2} \left(F_5^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) F_5^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon). \tag{28}$$

Of course it is simple to calculate $F_5^{(1)}(\epsilon)$ from the definitions (8) and (26),

$$F_5^{(1)}(\epsilon) = -\frac{1}{4} \sum_{i=1}^{5} \ln \left(\frac{s_i}{s_{i+1}} \right) \ln \left(\frac{s_{i-1}}{s_{i+2}} \right) + \frac{5\pi^2}{8} + \mathcal{O}(\epsilon). \tag{29}$$

Plugging this result into (28) leads to the following sharp prediction: If (5) holds at $\mathcal{O}(\epsilon^0)$, then the quantity defined in (27) must be equal to

$$F_5^{(2)}(\epsilon) = \frac{89}{1152}\pi^4 + \frac{11}{96}\pi^2(X - Y) + \frac{1}{32}(X - Y)^2 + \mathcal{O}(\epsilon), \tag{30}$$

where we have defined

$$X = \sum_{i=1}^{5} \ln^2 \left(\frac{s_i}{s_{i+1}} \right), \qquad Y = \sum_{i=1}^{5} \ln^2 \left(\frac{s_i}{s_{i+2}} \right). \tag{31}$$

Having checked the ABDK relation (5) as described above, the formula (30) for the finite remainder must also be true as a consequence. Nevertheless we found it useful to also check (30) directly via a more robust numerical procedure as follows. Supposing we didn't know (30) but wanted to check it numerically, we began with the ansatz that $F_5^{(2)}(0)$ should be a quadratic polynomial in $\ln^2$ of kinematical invariants. Now the right-hand side of (27) is

manifestly invariant under cyclic permutations $s_i \to s_{i+1}$, so $F_5^{(2)}(\epsilon)$ must have this symmetry as well. The objects X and Y defined in (31) are the only independent quantities with the right structure satisfying this symmetry. We therefore made the ansatz that $F_5^{(2)}(0)$ should be a quadratic polynomial in X and Y with six coefficients to be determined,

$$F_5^{(2)}(0) = a_1 \pi^4 + a_2 \pi^2 X + a_3 \pi^2 Y + a_4 X^2 + a_5 XY + a_6 Y^2. \tag{32}$$

The factors of π^2 were chosen conveniently so that all of the a_i were expected to be rational numbers. Indeed, we found that the rational numbers $a_1 = 89/1152$, $a_2 = -a_3 = 11/96$ and $a_4 = a_5 = -a_6/2 = 1/32$ were repeatedly obtained to a precision of 10^{-9} for various randomly generated values of the kinematical invariants, so we are quite confident that the finite remainder $F_5^{(2)}(0)$ is indeed given by (30). The robustness of this numerical check comes from the fact that we use numerical data to fix just six rational numbers.

IV. CONSISTENCY CHECKS ON THE $W_5^{(2)}(\epsilon)$ PIECE OF $M_5^{(2)}(\epsilon)$

In this section we study the structure of the proposed two-loop five-gluon amplitude given in [10]. First we show that $V_5^{(2)}(\epsilon)$ cannot possibly be equal to the full amplitude $M_5^{(2)}(\epsilon)$ by itself. This is done by computing a double-double cut in a two-particle channel to see that the $W_5^{(2)}(\epsilon)$ piece is needed. We then show that $W_5^{(2)}(\epsilon)$ has very mild IR behaviour, diverging only as $\mathcal{O}(\epsilon^{-1})$. This fact makes the presence of $W_5^{(2)}(\epsilon)$ consistent with the known IR [6, 7, 8] behaviour of $M_5^{(2)}(\epsilon)$.

A. Double-Double Cut

Unitarity cuts provide very powerful constraints on scattering amplitudes. At one loop, amplitudes in supersymmetric gauge theories are completely determined by their cuts [18, 19]. This fact is especially powerful if the basis of master integrals (after reduction has been carried out) is known. In particular, at one loop in $\mathcal{N} = 4$ SYM, the problem of computing any amplitude is reduced to that of computing tree amplitudes by using quadruple cuts [20].

At two loops, the basis of integrals is not known. In the case of $n = 4$ only double-box scalar integrals are needed to write the full amplitude [2]. This suggests that for $n = 5$ the basis should include the double boxes $I^{(2)a}$, $I^{(2)b}$ and $I^{(2)c}$ shown in Figure 2. In addition

one might expect pentagon-boxes, i.e, a pentagon joined to a box along one propagator. These are impossible for $n = 4$ but for $n = 5$ the integral $I^{(2)d}$ shown in Figure 2 is natural. One should also allow for the same classes of integrals with various tensor structures in the numerator.

The proposal of [10] for $M_5^{(2)}(\epsilon)$ is a linear combination of the integrals in Figure 2 with tensor structures taking the general form

$$A + B_\mu q^\mu + C_{\mu\nu} q^\mu q^\nu. \tag{33}$$

Our goal in this subsection is to show that the pieces in $V_5^{(2)}(\epsilon)$, as defined in (10), are not enough to be consistent with unitarity and that the proposal of [10] is just right to give the correct cuts.

Consider the double two-particle unitarity cut in the s_1 channel. Recall that unitarity cuts compute discontinuities across branch cuts of the amplitude. In this case, the double-double cut computes the discontinuity of the discontinuity of the amplitude across the branch cut in the s_1 channel. This is given by (we use the notation $P_{ij\cdots} \equiv k_i + k_j + \cdots$)

$$
\begin{aligned}
C_{12}^{12} = {}& \int \frac{d^D q}{(2\pi)^D}\, \delta^{(+)}(q^2)\delta^{(+)}((q + P_{345})^2) A^{\text{tree}}(-(q + P_{345})^-, 3^+, 4^+, 5^+, q^-) \\
&\times \int \frac{d^D \ell}{(2\pi)^D}\, \delta^{(+)}(\ell^2)\delta^{(+)}((\ell - P_{12})^2) A^{\text{tree}}((-q)^+, \ell^-, (-\ell + P_{12})^-, (q + P_{345})^+) \\
&\times A^{\text{tree}}((-\ell)^+, 1^-, 2^-, (\ell - P_{12})^+).
\end{aligned}
\tag{34}
$$

The integral over ℓ is easy to recognize as the cut in the s_1 channel of the one-loop amplitude $A^{(1)}((-q)^+, 1^-, 2^-, (q + P_{345})^+)$. Therefore[5],

$$
\begin{aligned}
C_{12}^{12} = {}& \int \frac{d^D q}{(2\pi)^D}\, \delta^{(+)}(q^2)\delta^{(+)}((q + P_{345})^2) A^{\text{tree}}((-q)^+, 1^-, 2^-, (q + P_{345})^+) \\
&\times A^{\text{tree}}(-(q + P_{345})^-, 3^+, 4^+, 5^+, q^-) \\
&\times \left[i \frac{e^{-\epsilon\gamma}}{(4\pi)^{2-\epsilon}} \right] s_1 (q - k_1)^2
\end{aligned}
\tag{35}
$$

[5] The strange prefactor in the last line converts from Feynman diagrams, which are normalized according to $\int d^D p/(2\pi)^D$, to the normalization (A2) for the scalar box diagram.

This expression can be further simplified by following the steps in section 5 of [18] where unitarity cuts of one-loop MHV amplitudes are studied. The first step is to use that $q^2 = 0$ and $(q + P_{345})^2 = 0$ to write

$$A^{\text{tree}}((-q)^+, 1^-, 2^-, (q + P_{345})^+) A^{\text{tree}}(-(q + P_{345})^-, 3^+, 4^+, 5^+, q^-) =$$
$$\frac{1}{2} A^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) \left[-\frac{\text{tr}_-(\slashed{q} \slashed{k}_1 (\slashed{q} - \slashed{P}_{12}) \slashed{k}_2)}{(q - k_1)^2 (q - k_1)^2} - \frac{\text{tr}_-(\slashed{q} \slashed{k}_5 (\slashed{q} - \slashed{P}_{12}) \slashed{k}_2)}{(q + k_5)^2 (q - k_1)^2} \right.$$
$$\left. - \frac{\text{tr}_-(\slashed{q} \slashed{k}_1 (\slashed{q} - \slashed{P}_{12}) \slashed{k}_3)}{(q - k_1)^2 (q + P_{45})^2} - \frac{\text{tr}_-(\slashed{q} \slashed{k}_5 (\slashed{q} - \slashed{P}_{12}) \slashed{k}_3)}{(q + k_5)^2 (q + P_{45})^2} \right], \quad (36)$$

where $\text{tr}_-(\bullet) = \frac{1}{2}\text{tr}((1 - \gamma_5)\bullet)$.

Now we expand all terms inside the bracket in such a way that the q-dependence is only in propagators or in $\text{tr}(\gamma_5...)$ terms. This immediately gives the result

$$C_{12}^{12} = \frac{1}{2} \left[i \frac{e^{-\epsilon\gamma}}{(4\pi)^{2-\epsilon}} \right]^2 A^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) s_1 \Big[(s_5 s_1 - \text{tr}(\gamma_5 \slashed{q} \slashed{k}_5 \slashed{k}_1 \slashed{k}_2)) I_{i \to i+2}^{(2)a}$$
$$+ (s_1 s_2 - \text{tr}(\gamma_5 \slashed{q} \slashed{k}_1 \slashed{k}_2 \slashed{k}_3)) I_{i \to i+3}^{(2)b} + (s_3 s_4 - \text{tr}(\gamma_5 \slashed{q} \slashed{k}_3 \slashed{k}_4 \slashed{k}_5)) I^{(2)d} \Big]. \quad (37)$$

We use an abbreviated but hopefully transparent notation here: On the first line, $I_{i \to i+2}^{(2)a}$ means the double-box integral $I^{(2)a}$ shown in Figure 2, but with k_i replaced by k_{i+2} (so that, for example, the massive leg has $k_3 + k_4$ instead of $k_1 + k_2$). Moreover, all three $\text{tr}(\gamma_5 \cdots)$ terms are meant to appear in the numerator of the corresponding integrand. Finally, the three diagrams $I^{(2)a}$, $I^{(2)b}$ and $I^{(2)d}$ appearing on the right-hand side of (37) should of course be double-double cut in the obvious way.

Note that except for the terms with $\text{tr}(\gamma_5...)$, (37) would be consistent[6] with $V_5^{(2)}(\epsilon)$. Luckily, the first two $\text{tr}(\gamma_5...)$ terms give no contribution at all. To see this note that the whole integral depends on only three independent momenta and by Lorentz invariance the trace must give zero. For example, the first term in (37) only depends on k_5, k_1, and k_2.

Now it is clear why the last term does not vanish. The reason is that the pentagon-box depends on four independent momenta. This shows that the $W_5^{(2)}(\epsilon)$ in $M_5^{(2)}(\epsilon)$ has to be there in order to be consistent with unitarity.

---

[6] Recall from equation (1) of [2] that $M_n^{(L)}(\epsilon)$ is normalized with a factor of $\left[2e^{-\epsilon\gamma}/(4\pi)^{2-\epsilon} \right]^{-L}$ relative to Feynman diagrams. Taking this into account turns the prefactor in (37) into $-\frac{1}{8}$, in agreement with (10).

B.   Infrared Behaviour of $W_5^{(2)}(\epsilon)$

Here we discuss the very special infrared behaviour that the parity odd term $W_5^{(2)}(\epsilon)$ possesses. The structure of IR singularities as poles in ϵ was first studied at two loops by Catani [7], who showed that the coefficients of the $1/\epsilon^4$, $1/\epsilon^3$ and $1/\epsilon^2$ terms in a two-loop amplitude $M_n^{(2)}(\epsilon)$ are given by[7]

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)M_n^{(1)}(2\epsilon) + \mathcal{O}(\epsilon^{-1}). \tag{38}$$

On the other hand, in this paper we have explicitly shown that (5)

$$V_5^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)M_n^{(1)}(2\epsilon) + \mathcal{O}(\epsilon^{-1}) \tag{39}$$

(we have replaced $V_5^{(1)}(\epsilon)$ by $M_5^{(1)}(\epsilon)$ on the right-hand side since the difference $W_5^{(1)}(\epsilon)$ only starts contributing to this equation at $\mathcal{O}(\epsilon^{-1})$). The only way to avoid a contradiction between (38) and (39) is if the remaining part $W_5^{(2)}(\epsilon) = M_5^{(2)}(\epsilon) - V_5^{(2)}(\epsilon)$ is $\mathcal{O}(\epsilon^{-1})$.

By Lorentz invariance it must be true that

$$W_5^{(2)}(\epsilon) = \mathrm{tr}(\gamma_5 \not{k}_1 \not{k}_2 \not{k}_3 \not{k}_4) w_5^{(2)}(\epsilon) \tag{40}$$

where $w_5^{(2)}(\epsilon)$ is a function of only the s_i kinematical invariants. To write $W_5^{(2)}(\epsilon)$ in the form (40) is straightforward using Feynman parameters where the presence of the original q^μ only leaves a Feynman parameter in the numerator. It is easy to construct a Mellin-Barnes representation for $w_5^{(2)}(\epsilon)$ (see appendix A.2). For example, one can take the integral (A21) with an additional factor of

$$\frac{\Gamma(-2-3\epsilon-\nu-z_1)\Gamma(-3-2\epsilon-z_1-z_3-z_4-z_5-z_6)}{\Gamma(-1-3\epsilon-\nu-z_1)\Gamma(-4-2\epsilon-z_1-z_3-z_4-z_5-z_6)} \tag{41}$$

in the integrand. By explicit computation using the program [11] we find the remarkable result that

$$w_5^{(2)}(\epsilon) = \frac{c_1}{\epsilon} + c_0 + \mathcal{O}(\epsilon) \tag{42}$$

where for example

$$
\begin{aligned}
c_1 \propto \int & \left[\prod_{i=1}^4 \frac{dz_i}{2\pi i}\Gamma(-z_i)\right] \left(\frac{s_1}{s_2}\right)^{z_1} \left(\frac{s_3}{s_2}\right)^{1+z_2} \left(\frac{s_4}{s_2}\right)^{1+z_3} \left(\frac{s_5}{s_2}\right)^{z_4} \\
& \times \Gamma(1+z_1+z_2)\Gamma(-1-z_1-z_2-z_3)\Gamma(1+z_1+z_3) \\
& \times \Gamma(-1-z_1-z_2-z_4)\Gamma(1+z_2+z_4)\Gamma(2+z_1+z_2+z_3+z_4). \tag{43}
\end{aligned}
$$

----

[7] This is adapting Catani's formula to the case of $\mathcal{N} = 4$ SYM.

This is a rather surprising result, because one would expect a generic two-loop tensor integral $q^\mu q^\nu$ to diverge as ϵ^{-4}. This integral is special because it contains the factor $q^\mu K_\mu$ in the numerator, where the vector $K_\mu = \epsilon_{\mu\nu\rho\sigma}(k_1 + k_2)^\nu k_3^\rho k_5^\sigma$ has zero inner product with many of the external momenta. One has to go all the way to $\mathcal{O}(\epsilon^{-1})$ before a term can appear with sufficiently complicated structure to survive being dotted with K_μ.

V.   CONCLUSION AND DISCUSSION

In this paper we have proven the striking result that $V_5^{(1)}(\epsilon)$ and $V_5^{(2)}(\epsilon)$ satisfy the ABDK relation (5), even though $V_5^{(1)}(\epsilon)$ is only part of the full one-loop amplitude $M_5^{(1)}(\epsilon)$, and $V_5^{(2)}(\epsilon)$ is only part of the full two-loop amplitude $M_5^{(2)}$. Given this result, it is easy to see that the full ABDK relation (1) holds if and only if the 'extra' pieces $W_5^{(L)}(\epsilon) \equiv M_5^{(L)}(\epsilon) - V_5^{(L)}(\epsilon)$ are related by

$$W_5^{(2)}(\epsilon) = V_5^{(1)}(\epsilon)W_5^{(1)}(\epsilon) + \mathcal{O}(\epsilon) = -\frac{5}{2\epsilon^2}(s_1 s_2 s_3 s_4 s_5)^{-\epsilon/5}\, W_5^{(1)}(\epsilon) + \mathcal{O}(\epsilon). \tag{44}$$

In [10] it was conjectured that

$$W_5^{(2)}(\epsilon) = -\frac{1}{8}\sum_{\text{cyclic}}\left[s_1 I^{(2)e}(\epsilon)\right]. \tag{45}$$

We have performed a couple of consistency checks on this conjecture. First, we studied some unitarity cuts that explicitly show the presence of the $W_5^{(2)}(\epsilon)$ term inside $M_5^{(2)}(\epsilon)$. We also remarked that (45) has a very mild leading IR singularity $\mathcal{O}(\epsilon^{-1})$, which is consistent with (44) since $W_5^{(1)}(\epsilon)$ is $\mathcal{O}(\epsilon)$.

Finally, it is interesting to compare the formula for $M_5^{(2)}(\epsilon)$ in (9) with the results obtained by using the recent technique of octa-cuts introduced in [21]. The basic idea of the octa-cut technique is that at two loops one can cut all propagators belonging to a box, i.e. perform a quadruple cut [20] inside a double box or a pentagon-box and get from the Jacobian a new propagator. The new propagator is just right so as to produce a new effective one-loop box or pentagon integral. The remaining one-loop object can then be cut four times to localize the final loop integration momentum. In [21], the coefficients of all double-boxes for the five-gluon amplitude were computed. Perfect agreement is found with (9) for the $I^{(2)a}$ and $I^{(2)b}$ integrals. However, there seem to be a discrepancy for the $I^{(2)c}$ integral, which is missing from (9) but should have a nonzero coefficient according to [21].

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The solution to this puzzle is to realize that $I^{(2)d}$ secretly contains an $I^{(2)c}$ integral. To see this, expand the numerator $(q - k_1)^2 = q^2 - 2k_1 \cdot q$; the first term q^2 cancels a propagator and produces precisely the $I^{(2)c}$ integral[8]. We then find perfect agreement with the octa-cut calculation of the $I^{(2)c}$ coefficient in [21][9].

This preliminary analysis shows that (9) satisfies several non trivial constraints. As a future direction, it would be very desirable to make a more detailed analysis of all unitarity cuts of $M_5^{(2)}(\epsilon)$. It would also be interesting to compute $W_5^{(1)}(\epsilon)$ in order to determine whether the ansatz (45) satisfies (44).

APPENDIX A: EVALUATION OF INTEGRALS

This appendix contains many of the technical details referred to in the text. The one- and two-loop integrals we need to study are shown in Figures 1 and 2. Our convention is that each loop momentum integral comes with a normalization factor of

$$- i e^{\epsilon \gamma} \pi^{-D/2} \int d^D p, \tag{A1}$$

with $D = 4 - 2\epsilon$. This is the conventional normalization for studying iteration relations and agrees with that used in writing (1), although it differs slightly from the normalization of amplitudes which should appear in the actual S-matrix (see [2] for details). As an illustration, the one-loop one-mass scalar box integral of Figure 1 is defined to be

$$I^{(1)}(\epsilon) \equiv \quad \begin{array}{c} k_3 \qquad k_4 \\ \Box \\ k_1 + k_2 \qquad k_5 \end{array} \quad = -i e^{\epsilon \gamma} \pi^{-D/2} \int d^D p \; \frac{1}{p^2 (p - k_5)^2 (p + k_3 + k_4)^2 (p + k_4)^2}. \tag{A2}$$

We begin in subsection A.1 by recording our results for the various integrals, before detailing the method by which they were obtained. In subsection A.2 we present Mellin-Barnes representations for the new integrals $I^{(2)c}$ and $I^{(2)d}$. Finally in subsection A.3 we

---

[8] Actually, a given $I^{(2)d}$ integral can be decomposed in two different ways but we choose as the canonical decomposition the one just described.

[9] Slightly more subtle is to prove that (9) has the correct octa-cut in all "channels". A more detailed analysis which is beyond the scope of this paper shows that is the case.

explain how we obtained the results of A.1 by expressing everything in terms of a small number of basic Mellin-Barnes-type integrals.

1. Results

In what follows we use the notation

$$s_i \equiv -(k_i + k_{i+1})^2, \qquad L_i \equiv \ln s_i \tag{A3}$$

and the functions

$$A(a) = -H_{1,1}(1-a) - H_{0,1}(1-a) - \frac{\pi^2}{6}, \tag{A4}$$

$$A_1(a) = -H_{1,1,1}(1-a) + H_{0,0,1}(1-a) - \frac{\pi^2}{6}H_1(1-a), \tag{A5}$$

$$A_2(a) = H_{1,1,1}(1-a) + H_{1,0,1}(1-a) + H_{0,1,1}(1-a) + H_{0,0,1}(1-a) + \zeta(3), \tag{A6}$$

$$A_3(a) = H_{1,1,1}(1-a) + H_{1,0,1}(1-a) + \frac{\pi^2}{6}H_1(1-a) + 2\zeta(3), \tag{A7}$$

and

$$\begin{aligned}
F(a,b) = &\ \frac{\pi^2}{6}H_1\left(1-\frac{1}{a}\right) - H_{0,0,1}\left(1-\frac{1}{a}\right) - H_{0,0,1}\left(\frac{1-a-b}{1-a}\right) + \frac{1}{2}H_{0,0,1}\left(\frac{1-a-b}{(1-a)(1-b)}\right) \\
&+ H_{0,1,1}\left(\frac{1-a-b}{1-a}\right) - \frac{1}{2}H_{0,1,1}\left(\frac{1-a-b}{(1-a)(1-b)}\right) + H_{1,1,1}\left(1-\frac{1}{a}\right) \\
&- H_{0,1}\left(\frac{1-a-b}{1-a}\right)\ln(1-a) + H_{0,1}\left(\frac{1-a-b}{(1-a)(1-b)}\right)\ln(1-a) - \frac{\pi^2}{6}\ln(a) \\
&- H_{0,1}\left(1-\frac{1}{a}\right)\ln(a) - H_{1,1}\left(1-\frac{1}{a}\right)\ln(a) + \frac{1}{2}\ln^2(1-a)\ln(1-b) \\
&+ (a \leftrightarrow b).
\end{aligned} \tag{A8}$$

We use here standard conventions for harmonic polylogarithm functions (see for example the very useful program [22]). It is very likely that the expression (A8) can be simplified using various harmonic polylogarithm identities. However, as we have emphasized, it is possible to explicitly verify (17) without knowing the precise formula (A8). It turns out that as long as $F(a,b)$ is symmetric in a and b, which we have made manifest in (A8), then it automatically drops out of (17).

The one-loop one-mass scalar box integral $I^{(1)}$ was evaluated to all orders in ϵ in [14]. In our notation it is given by

$$I^{(1)}(\epsilon) = \frac{(s_3 s_4)^{-1-\epsilon}}{(s_1)^{-\epsilon}}\left[\frac{2}{\epsilon^2} + I_0^{(1)} + \epsilon I_{+1}^{(1)} + \mathcal{O}(\epsilon^2)\right] \tag{A9}$$

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with

$$I_0^{(1)} = 2A\left(\frac{s_1}{s_3}\right) + 2A\left(\frac{s_1}{s_4}\right) + \frac{\pi^2}{6},$$

$$I_{+1}^{(1)} = -2F\left(\frac{s_3}{s_1}, \frac{s_4}{s_1}\right) - 2\ln\left(\frac{s_1}{s_3}\right)A_1\left(\frac{s_1}{s_3}\right) - 2\ln\left(\frac{s_1}{s_4}\right)A_1\left(\frac{s_1}{s_4}\right) - \frac{14}{3}\zeta(3). \quad \text{(A10)}$$

The two-loop integral $I^{(2)a}$ was evaluated through $\mathcal{O}(\epsilon^0)$ in [16]. In our notation it is given by

$$I^{(2)a}(\epsilon) = \frac{(s_3)^{-1-2\epsilon}(s_4)^{-2-2\epsilon}}{(s_1)^{-2\epsilon}}\left[-\frac{1}{\epsilon^4} + \frac{I_{-2}^{(2)a}}{\epsilon^2} + \frac{I_{-1}^{(2)a}}{\epsilon} + \mathcal{O}(\epsilon^0)\right], \quad \text{(A11)}$$

with

$$I_{-2}^{(2)a} = -A\left(\frac{s_1}{s_3}\right) - 3A\left(\frac{s_1}{s_4}\right) - \frac{5\pi^2}{12}, \quad \text{(A12)}$$

$$I_{-1}^{(2)a} = 2F\left(\frac{s_3}{s_1}, \frac{s_4}{s_1}\right) + 2A_1\left(\frac{s_1}{s_3}\right) + 5A_1\left(\frac{s_1}{s_4}\right) - 2A_2\left(\frac{s_1}{s_3}\right) + A_2\left(\frac{s_1}{s_4}\right) - 3A_3\left(\frac{s_1}{s_3}\right)$$

$$+A_3\left(\frac{s_1}{s_4}\right) + 2\ln\left(\frac{s_1}{s_3}\right)A\left(\frac{s_1}{s_3}\right) + 4\ln\left(\frac{s_1}{s_4}\right)A\left(\frac{s_1}{s_4}\right) + \frac{25}{6}\zeta(3) \quad \text{(A13)}$$

The two-loop integral $I^{(2)b}$ is clearly given simply by

$$I^{(2)b}(\epsilon)(s_1, s_3, s_4) = I^{(2)a}(\epsilon)(s_1, s_4, s_3). \quad \text{(A14)}$$

The two-loop integral $I^{(2)c}$ has not appeared in the literature. We evaluate this integral[10] only through $\mathcal{O}(\epsilon^{-2})$, since that is sufficient to rule out its appearance in the ABDK relation (see section III.A). We find

$$I^{(2)c}(\epsilon) = \frac{(s_1 s_3)^{-1-2\epsilon/3}(s_2)^{-1-2\epsilon}}{(s_4 s_5)^{-2\epsilon/3}}\left[-\frac{9}{4\epsilon^4} + \frac{I_{-2}^{(2)c}}{\epsilon^2} + \mathcal{O}(\epsilon^{-1})\right], \quad \text{(A15)}$$

with

$$I_{-2}^{(2)c} = -A\left(\frac{s_4}{s_1}\right) - 2A\left(\frac{s_4}{s_2}\right) - 2A\left(\frac{s_5}{s_2}\right) - A\left(\frac{s_5}{s_3}\right) + \frac{1}{2}\ln^2\left(\frac{s_1}{s_4}\right) + \frac{1}{2}\ln^2\left(\frac{s_3}{s_5}\right) + \frac{5\pi^2}{24}. \quad \text{(A16)}$$

Finally we have the integral $I^{(2)d}$, which also has not appeared in the literature. We find

$$I^{(2)d}(\epsilon) = \frac{(s_2 s_5)^{-2\epsilon/3}(s_3 s_4)^{-1-\epsilon/3}}{s_1}\left[-\frac{3}{\epsilon^4} + \frac{I_{-2}^{(2)d}}{\epsilon^2} + \frac{I_{-1}^{(2)d}}{\epsilon} + \mathcal{O}(\epsilon^0)\right], \quad \text{(A17)}$$

---

[10] MS and AV are grateful to R. Roiban for collaboration on early attempts to evaluate this integral.

where

$$
\begin{aligned}
I_{-2}^{(2)d} = {}& -3A\left(\frac{s_1}{s_3}\right) - 2A\left(\frac{s_4}{s_2}\right) - A\left(\frac{s_3}{s_1}\right) - A\left(\frac{s_4}{s_1}\right) - 3A\left(\frac{s_1}{s_4}\right) - 2A\left(\frac{s_3}{s_5}\right) - \frac{\pi^2}{6} \\
& - \frac{4}{3}\ln^2\left(\frac{s_2}{s_5}\right) + \frac{1}{6}\ln^2\left(\frac{s_3}{s_4}\right) - \frac{4}{3}(L_2L_3 + L_4L_5) + \frac{2}{3}(L_2 + L_3)(L_4 + L_5), \quad \text{(A18)}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{-1}^{(2)d} = {}& 2F\left(\frac{s_1}{s_3}, \frac{s_5}{s_3}\right) + 2F\left(\frac{s_3}{s_1}, \frac{s_4}{s_1}\right) + 2F\left(\frac{s_2}{s_4}, \frac{s_1}{s_4}\right) \\
& + 6A_1\left(\frac{s_1}{s_3}\right) + 2A_1\left(\frac{s_3}{s_1}\right) - A_1\left(\frac{s_1}{s_4}\right) + A_1\left(\frac{s_4}{s_1}\right) \\
& + A_2\left(\frac{s_1}{s_3}\right) - A_2\left(\frac{s_3}{s_1}\right) + 6A_2\left(\frac{s_1}{s_4}\right) - 2A_2\left(\frac{s_4}{s_1}\right) - 2A_2\left(\frac{s_4}{s_2}\right) - 2A_2\left(\frac{s_3}{s_5}\right) \\
& + 3A_3\left(\frac{s_1}{s_3}\right) + A_3\left(\frac{s_3}{s_1}\right) + A_3\left(\frac{s_4}{s_1}\right) + 2A_3\left(\frac{s_4}{s_2}\right) + 2A_3\left(\frac{s_3}{s_5}\right) - 4A_3\left(\frac{s_5}{s_2}\right) \\
& + (3L_1 - 2L_2 - 4L_3 + L_4 + 2L_5)A\left(\frac{s_1}{s_3}\right) - \frac{1}{3}(3L_1 + 2L_2 - 2L_3 + L_4 - 4L_5)A\left(\frac{s_3}{s_1}\right) \\
& - \frac{1}{3}(3L_1 - 4L_2 + L_3 - 2L_4 + 2L_5)A\left(\frac{s_4}{s_1}\right) + (2L_2 + L_3 - L_4 - 2L_5)A\left(\frac{s_1}{s_4}\right) \\
& + \frac{2}{3}(L_2 + 2L_3 + 2L_4 - 5L_5)A\left(\frac{s_4}{s_2}\right) - \frac{2}{3}(5L_2 - 2L_3 - 2L_4 - L_5)A\left(\frac{s_3}{s_5}\right) \\
& + \frac{2}{9}\pi^2(L_2 + 2L_3 + 2L_4 - 5L_5) + \frac{1}{27}(2L_2 + L_3 + L_4 - 2L_5)^3 \\
& - \frac{2}{3}(L_2L_3^2 + L_2^2L_4 + L_2L_3L_4 + L_3L_5^2) + \frac{10}{9}L_2(L_3 + L_4)L_5 + \frac{4}{9}L_3^2L_5 \\
& + \frac{2}{9}(L_3L_4 - L_4^2 - L_2^2)L_5 - \frac{2}{27}L_5^3 + 14\zeta(3) \quad \text{(A19)}
\end{aligned}
$$

2. Mellin-Barnes Representations

In order to evaluate these integrals we used the technology of Mellin-Barnes (MB) integral representations (see [23] for a thorough review). It is straightforward to develop a MB representation for all of the above integrals (see [17] for a fairly general treatment). For example, the new integrals $I^{(2)c}$ and $I^{(2)d}$ admit the (highly non-unique) MB representations

$$
\begin{aligned}
I^{(2)c}(\epsilon) = {}& -\frac{e^{2\epsilon\gamma}}{s_5^{3+2\epsilon}\Gamma(-2\epsilon)}\int\left[\prod_{i=1}^{7}\frac{dz_i}{2\pi i}\Gamma(-z_i)\right]\left(\frac{s_1}{s_5}\right)^{z_4}\left(\frac{s_2}{s_5}\right)^{z_5}\left(\frac{s_3}{s_5}\right)^{z_3+z_7}\left(\frac{s_4}{s_5}\right)^{z_6} \\
& \times \frac{\Gamma(-1 - \epsilon - z_1 - z_2)}{\Gamma(1 - z_1)\Gamma(-z_2)\Gamma(-1 - 3\epsilon - z_3)}\Gamma(1 + z_2)\Gamma(-1 - \epsilon - z_3) \\
& \times \Gamma(1 + z_1 + z_3)\Gamma(-1 - \epsilon + z_1 + z_2 - z_4 - z_5 - z_6)\Gamma(1 + z_4 + z_6) \\
& \times \Gamma(-z_2 + z_5 + z_6)\Gamma(-2 - 2\epsilon - z_3 - z_4 - z_6 - z_7) \\
& \times \Gamma(1 - z_1 + z_4 + z_7)\Gamma(3 + 2\epsilon + z_3 + z_4 + z_5 + z_6 + z_7) \quad \text{(A20)}
\end{aligned}
$$

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and

$$I^{(2)d}(\epsilon) = \lim_{\nu \to -1} (-1)^\nu \frac{e^{2\epsilon\gamma}}{\Gamma(-2\epsilon)} \int \left[\prod_{i=1}^{7} \frac{dz_i}{2\pi i} \Gamma(-z_i) \right] \left(\frac{s_1}{s_2} \right)^{z_1+z_5} \left(\frac{s_3}{s_2} \right)^{z_6} \left(\frac{s_4}{s_2} \right)^{z_4} \left(\frac{s_5}{s_2} \right)^{z_7}$$

$$\times \frac{\Gamma(-1-\epsilon-z_1)\Gamma(1+z_1+z_2)\Gamma(-1-\epsilon-z_2-z_3)}{\Gamma(-2-3\epsilon-\nu-z_1)\Gamma(1-z_2)\Gamma(\nu-z_3)\Gamma(3+\epsilon+z_1+z_2+z_3)}$$

$$\times \Gamma(1+z_3)\Gamma(2+\epsilon+z_1+z_2+z_3)\Gamma(1-z_2+z_4+z_5)$$

$$\times \Gamma(-4-2\epsilon-z_1-z_3-z_4-z_5-z_6)\Gamma(3+\epsilon+z_1+z_2+z_3+z_5+z_6)$$

$$\times \Gamma(-3-2\epsilon-\nu-z_1-z_5-z_6-z_7)\Gamma(1+z_6+z_7)$$

$$\times \Gamma(4+2\epsilon+\nu+z_1+z_4+z_5+z_6+z_7). \tag{A21}$$

A few comments are in order. In each case the contours of integration for the z_i variables can be taken to be lines parallel to the imaginary axis, as long as $\mathrm{Re}(z_i)$ are chosen such that the arguments of all Γ functions have a positive real part. In the expression (A21), the quantity ν originates as the power of the factor $(q - k_1)^2$ in the denominator of the Feynman integral. Since we want this factor to end up in the numerator, we want to take $\nu \to -1$. This limit must be defined by analytic continuation, since it is not possible to satisfy all of the constraints on the z_i contours if one takes $\nu = -1$ at the start.

3. Building Blocks

A very efficient package for developing the ϵ series expansion of a general Mellin-Barnes integral has recently been developed by Czakon [11]. Using this package, we can easily read off the values of the desired integrals, order by order in ϵ, in terms of some basic building blocks. Specifically, we find that for all of the integrals considered in this paper, only the following two integral structures appear through order ϵ^{-2}:

$$A(a) = \int \frac{dz}{2\pi i} \, a^z \, \Gamma^2(-z)\Gamma(z)\Gamma(1+z), \tag{A22}$$

$$B(a) = \int \frac{dz}{2\pi i} \, a^{z+1} \Gamma(-1-z)\Gamma(-z)\Gamma^2(1+z). \tag{A23}$$

The integration variable z in each of these expressions can be taken to run along a line parallel to the imaginary axis from $-\frac{1}{4} - i\infty$ to $-\frac{1}{4} + i\infty$. These integrals can be evaluated explicitly using standard techniques. For $a > 0$ we find $A(a)$ as written in (A4), and

$$B(a) = H_{0,1}(1-a) - \frac{\pi^2}{6}. \tag{A24}$$

We have chosen to express the argument of each dilogarithm as $1 - a$ so that the branch cuts of the integrals (A22) and (A23) for $a \in (-\infty, 0)$ manifestly map on to the branch cuts of the harmonic polylogarithms on the right-hand side.

Although we have written explicit formulas for $A(a)$ and $B(a)$ for the reader's benefit, one of the points we want to emphasize in this paper (following [15]) is that it is more efficient to check ABDK-type relations 'under the integral sign.' For example, instead of evaluating the integrals (A22) and (A23) explicitly, it is simpler to notice that by taking $z \to -1 - z$ inside the integral we see immediately that

$$A(a) = B(1/a). \tag{A25}$$

This procedure of identifying the basic Mellin-Barnes building blocks such as (A22) and (A23), and then deriving various identities between them, can be carried out (with increasing complication, of course) at each order in ϵ. At $\mathcal{O}(\epsilon^{-1})$ a number of new integrals, including double-integrals, start to appear. Using various tricks 'under the integral sign,' they can all be expressed without too much difficulty in terms of the basic integrals

$$A_1(a) = \int \frac{dz}{2\pi i} \, a^z \, \Gamma^2(-z)\Gamma(z)\Gamma(1+z)(\psi(z) + \gamma), \tag{A26}$$

$$A_2(a) = \int \frac{dz}{2\pi i} \, a^z \, \Gamma^2(-z)\Gamma(z)\Gamma(1+z)(\psi(-z) + \gamma), \tag{A27}$$

$$A_3(a) = \int \frac{dz}{2\pi i} \, a^z \, \Gamma^2(-z)\Gamma(z)\Gamma(1+z)\frac{1}{z}, \tag{A28}$$

$$F(a,b) = \int \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i} \, a^{1+z_2}b^{1+z_1}\Gamma(-1-z_1)\Gamma(-z_1)\Gamma(1+z_1)\Gamma(-1-z_2)$$
$$\times \Gamma(-z_2)\Gamma(1+z_2)\Gamma(2+z_1+z_2). \tag{A29}$$

We have tabulated the explicit evaluations of these integrals in subsection A.1, but we emphasize that it is possible to verify the ABDK relation (5) through $\mathcal{O}(\epsilon^{-1})$ without needing to evaluate these integrals, due to the identities (18).

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[1] E. Witten, "Perturbative gauge theory as a string theory in twistor space," Commun. Math. Phys. **252**, 189 (2004) [arXiv:hep-th/0312171].

[2] C. Anastasiou, Z. Bern, L. J. Dixon and D. A. Kosower, "Planar amplitudes in maximally supersymmetric Yang-Mills theory," Phys. Rev. Lett. **91**, 251602 (2003) [arXiv:hep-th/0309040].

[3] Z. Bern, V. Del Duca and C. R. Schmidt, "The infrared behavior of one-loop gluon amplitudes at next-to-next-to-leading order," Phys. Lett. B **445**, 168 (1998) [arXiv:hep-ph/9810409].

[4] D. A. Kosower, "All-order collinear behavior in gauge theories," Nucl. Phys. B **552**, 319 (1999) [arXiv:hep-ph/9901201].

[5] D. A. Kosower and P. Uwer, "One-loop splitting amplitudes in gauge theory," Nucl. Phys. B **563**, 477 (1999) [arXiv:hep-ph/9903515].

[6] L. Magnea and G. Sterman, "Analytic Continuation Of The Sudakov Form-Factor In QCD," Phys. Rev. D **42**, 4222 (1990).

[7] S. Catani, "The singular behaviour of QCD amplitudes at two-loop order," Phys. Lett. B **427**, 161 (1998) [arXiv:hep-ph/9802439].

[8] G. Sterman and M. E. Tejeda-Yeomans, "Multi-loop amplitudes and resummation," Phys. Lett. B **552**, 48 (2003) [arXiv:hep-ph/0210130].

[9] Z. Bern, L. J. Dixon and V. A. Smirnov, "Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond," Phys. Rev. D **72**, 085001 (2005) [arXiv:hep-th/0505205].

[10] Z. Bern, J. Rozowsky and B. Yan, "Two-loop N = 4 supersymmetric amplitudes and QCD," arXiv:hep-ph/9706392.

[11] M. Czakon, "Automatized analytic continuation of Mellin-Barnes integrals," arXiv:hep-ph/0511200.

[12] Z. Bern, L. J. Dixon and D. A. Kosower, "The Five gluon amplitude and one loop integrals," arXiv:hep-ph/9212237.

[13] Z. Bern, L. J. Dixon and D. A. Kosower, "Dimensionally regulated one loop integrals," Phys. Lett. B **302**, 299 (1993) [Erratum-ibid. B **318**, 649 (1993)] [arXiv:hep-ph/9212308].

[14] Z. Bern, L. J. Dixon and D. A. Kosower, "Dimensionally regulated pentagon integrals," Nucl. Phys. B **412**, 751 (1994) [arXiv:hep-ph/9306240].

[15] F. Cachazo, M. Spradlin and A. Volovich, "Hidden beauty in multiloop amplitudes," arXiv:hep-th/0601031.

[16] V. A. Smirnov, "Analytical result for dimensionally regularized massless master double box with one leg off shell," Phys. Lett. B **491**, 130 (2000) [arXiv:hep-ph/0007032].

[17] C. Anastasiou and A. Daleo, "Numerical evaluation of loop integrals," arXiv:hep-ph/0511176.

[18] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, "One loop n point gauge theory amplitudes, unitarity and collinear limits," Nucl. Phys. B **425**, 217 (1994) [arXiv:hep-ph/9403226].

[19] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, "Fusing gauge theory tree amplitudes into loop amplitudes," Nucl. Phys. B **435**, 59 (1995) [arXiv:hep-ph/9409265].

[20] R. Britto, F. Cachazo and B. Feng, "Generalized unitarity and one-loop amplitudes in $\mathcal{N} = 4$ super-Yang-Mills," Nucl. Phys. B **725**, 275 (2005) [arXiv:hep-th/0412103].

[21] E. I. Buchbinder and F. Cachazo, "Two-loop amplitudes of gluons and octa-cuts in $\mathcal{N} = 4$ super Yang-Mills," JHEP **0511**, 036 (2005) [arXiv:hep-th/0506126].

[22] D. Maitre, "`HPL`, a Mathematica implementation of the harmonic polylogarithms," arXiv:hep-ph/0507152.

[23] V. A. Smirnov, "Evaluating Feynman Integrals," Springer tracts in modern physics, **211** (Springer, Berlin, Heidelberg, 2004).